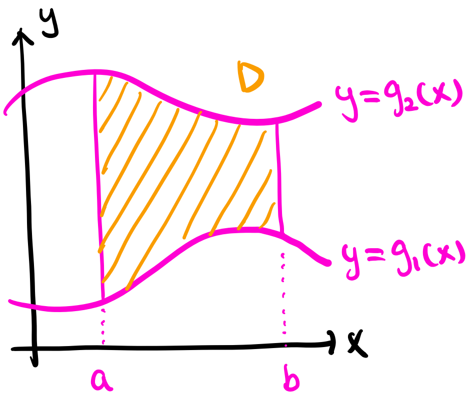
15.2. Double integrals over general regions: definitions

Def Let $f(x,y)$ be a continuous function on a domain D.

(1) D is of type I if it is of the form

$$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(y)\},$$

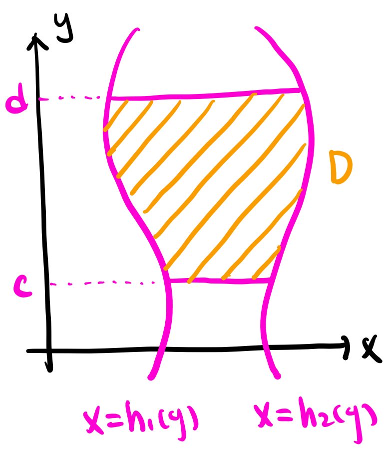


over which the integral of $f(x,y)$ is given by

$$\iint_D f(x,y)\,dA := \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(2) D is of type II if it is of the form

$$D = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$



over which the integral of $f(x,y)$ is given by

$$\iint_D f(x,y)\,dA := \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy.$$

<u>Note</u> (1) The order of integration is important for general domains

(2) The outer integral must have constant bounds.

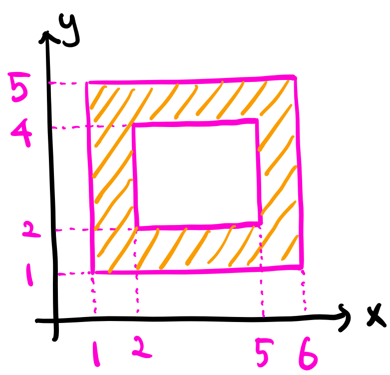
* Otherwise, the integral would not yield a constant value.

(3) Many domains are of both type I and type II, as we will soon see. For such domains, it is often important to choose the type that yields the simpler integral.

(4) Some domains are of neither type I nor type II.

not important in Math 215.

e.g.



$2 \le x \le 5$: $1 \le y \le 2$ or $4 \le y \le 5$

$\Rightarrow$ not type I

$2 \le y \le 4$: $1 \le x \le 2$ or $5 \le x \le 6$

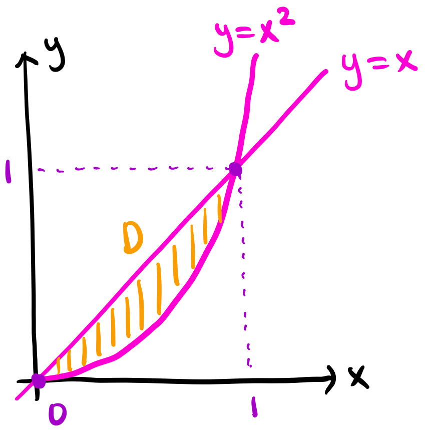
$\Rightarrow$ not type II.

However, in practice all such domains can be split into subdomains of type I or type II.

(5) As long as you can describe a given domain with the correct bounds, you do not need to know whether it is of type I or type II.

Ex Evaluate $\iint_D x+2y\, dA$ where D is the region enclosed by the curves $y=x$ and $y=x^2$.

Sol



Intersection : $y=x$ and $y=x^2$

$\Rightarrow x=x^2 \Rightarrow x=0,1$

$\Rightarrow (x,y)=(0,0),(1,1)$

$D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ x^2\le y\le x\}$ (type I)

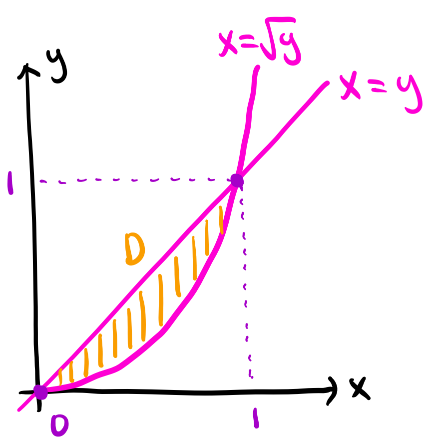
$$\iint_D x+2y\, dA = \int_0^1\int_{x^2}^{x} x+2y\, dy\, dx = \int_0^1 xy+y^2\Big|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 2x^2-x^3-x^4\, dx = \left(\frac{2}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5\right)\Big|_{x=0}^{x=1}$$

$$= \boxed{\frac{13}{60}}$$

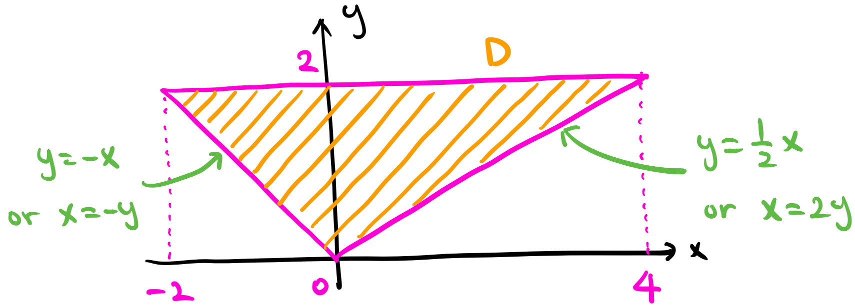
Note D is also of type II :

$D=\{(x,y)\in\mathbb{R}^2 : 0\le y\le 1,\ y\le x\le\sqrt{y}\}$.



Ex Evaluate $\iint_D 2xy\,dA$ where D is the triangular region with vertices at $(0,0)$, $(-2,2)$, and $(4,2)$.

Sol



$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2,\ -y \le x \le 2y\}$ (type II)

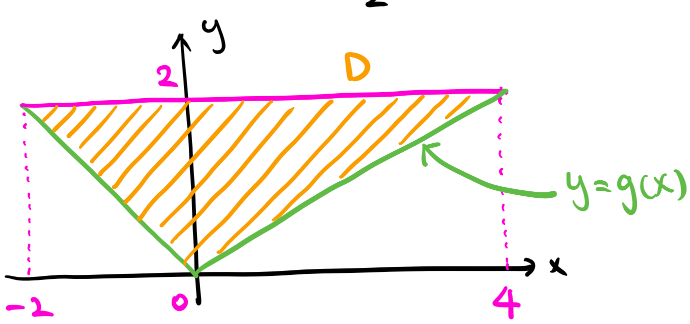
$$\iint_D 2xy\,dA = \int_0^2 \int_{-y}^{2y} 2xy\,dx\,dy = \int_0^2 x^2 y \Big|_{x=-y}^{x=2y} dx$$

$$= \int_0^2 3y^3\,dy = \frac{3}{4}y^4 \Big|_{y=0}^{y=2} = \boxed{12}$$

Note D is also of type I:

$D = \{(x,y) \in \mathbb{R}^2 : -2 \le x \le 4,\ g(x) \le y \le 2\}$

with $g(x) = \begin{cases} -x & \text{for } x < 0 \\ \frac{1}{2}x & \text{for } x \ge 0 \end{cases}$



$$\Rightarrow \iint_D 2xy\,dA = \int_{-2}^{4} \int_{g(x)}^{2} 2xy\,dy\,dx.$$

This integral is more difficult to compute than the one we used above.

Ex Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$.

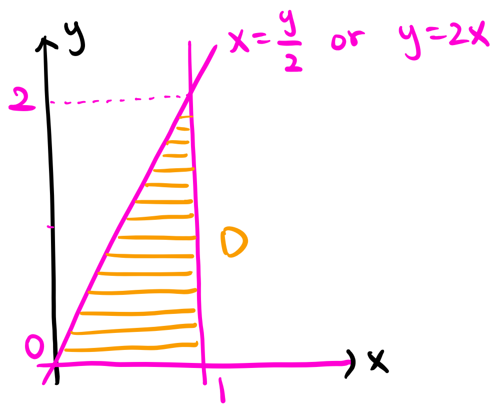
Sol We cannot compute the inner integral as given.

$*$ The antiderivative of e^{x^2} has no simple formula.

Idea: Switch the order of integration by switching the type of the domain.

The given integral describes the domain as

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2, \frac{y}{2} \le x \le 1\}$. (type II)



$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 2x\}$. (type I)

$$\Rightarrow \iint_D e^{x^2}\,dA = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 2x e^{x^2}\,dx$$

$$= e^{x^2}\Big|_{x=0}^{x=1} = \boxed{e-1}$$